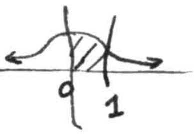
Name: Solutions Section: ____________

Net Area and Integrals

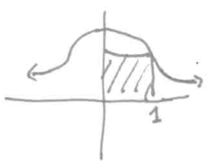


1. The integral $\int_a^b e^{-x^2}\,dx$ is important in statistics,[1] but it is infamously hard to compute. Many statistics textbooks include a table which lists the value of the integral for different values of a and b. We will use Reimann Sums to generate one of these approximations.

   (a) Express the integral $\int_0^1 e^{-x^2}\,dx$ as the limit of its Right Reimann Sums.

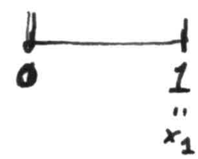
   $$\int_0^1 e^{-x^2}dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{-(x_i)^2}\cdot\Delta x$$ where $\Delta x = \frac{1-0}{n}$ and $x_i = a + i\cdot\Delta x$

   (b) Approximate $\int_0^1 e^{-x^2}\,dx$ using Right Sums and $n = 1$. Use a calculator to simplify.



   $R_1 = f(x_1)\cdot\Delta x$ $\quad \Delta x = \frac{1-0}{1}$ $\quad x_1 = 1$

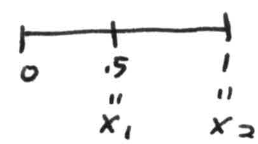
   $= e^{-(1)^2}\cdot 1 \approx .36788$



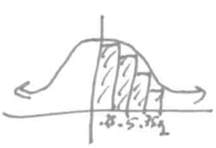
   (c) Approximate $\int_0^1 e^{-x^2}\,dx$ using Right Sums and $n = 2$. Use a calculator to simplify.

   $R_2 = f(x_1)\Delta x + f(x_2)\Delta x$ $\quad \Delta x = \frac{1-0}{2} = .5$

   $= e^{-(.5)^2}(.5) + e^{-(1)^2}(.5) \approx .57334$



   (d) Approximate $\int_0^1 e^{-x^2}\,dx$ using Right Sums and $n = 4$. Use a calculator to simplify.

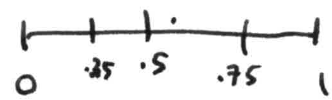


   $R_4 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$

   $= e^{-(.25)^2}(.25) + e^{-(.5)^2}(.25) + e^{-(.75)^2}(.25) + e^{-(1)^2}(.25)$

   $\approx .66397$

   $\Delta x = \frac{1-0}{4} = .25$



   $\Delta x = \frac{1-0}{8} = .125$

   (e) Approximate $\int_0^1 e^{-x^2}\,dx$ using Right Sums and $n = 8$. Use a calculator to simplify.

   $R_8 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x + f(x_7)\Delta x + f(x_8)\Delta x$

   $= (.125)\left[e^{-(.125)^2} + e^{-(.25)^2} + e^{-(.375)^2} + e^{-(.5)^2} + e^{-(.625)^2} + e^{-(.75)^2} + e^{-(.875)^2} + e^{-(1)^2}\right]$

   $\approx .70636$

   (f) How do these compare to the correct value of $\int_0^1 e^{-x^2}\,dx = .7468241\ldots$?

   they get closer as n gets bigger.

[1]This and other similar integrals are needed to compute the probability of events that follow a normal distribution. See, for example, http://en.wikipedia.org/wiki/Standard_normal_distribution#Cumulative_distribution.

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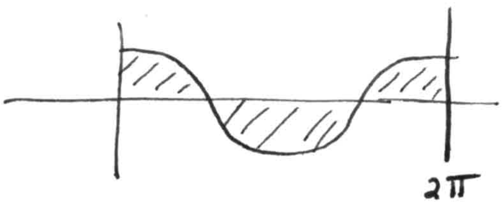
2. What is the graphical meaning of $\int_0^{2\pi} \cos(x)\,dx$? Compute this area geometrically.

the net area



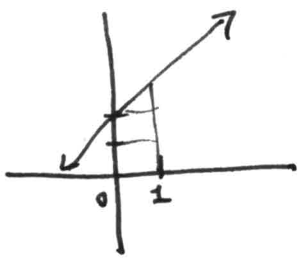
These areas cancel

so

integral = net area = 0

3. What is the graphical meaning of $\int_0^1 x + 2\,dx$? Compute this area geometrically.

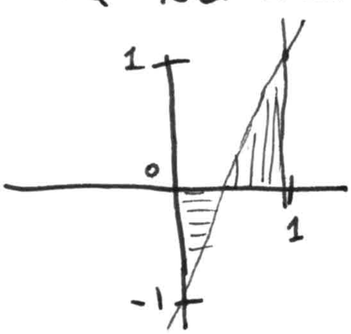
the net area below



net area $= 1 + 1 + \frac{1}{2} = \frac{5}{2}$

4. What is the graphical meaning of $\int_0^1 2x - 1\,dx$? Compute this area geometrically.

the net area $= \left(-\frac{1}{4}\right) + \left(\frac{1}{4}\right) = 0$

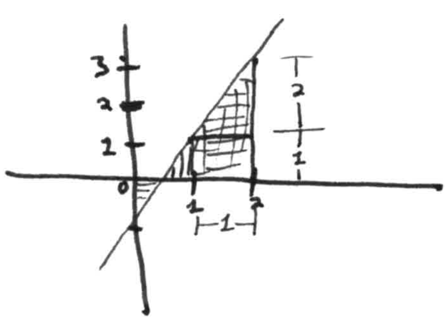


5. What is the graphical meaning of $\int_0^2 2x - 1\,dx$? Compute this area geometrically.

the net area

net area $= \left(\frac{-1}{4}\right) + \left(\frac{1}{4}\right) + 1 + \frac{1}{2}(2)(1)$

$= 1 + 1 = 2$



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6. Answer the following **Yes or No**

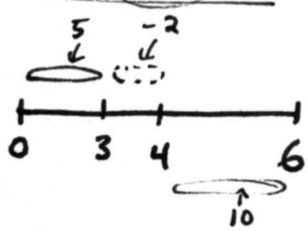
   (a) Can you distribute integrals across sums?
   i.e. does $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$? Yes No

   (b) Can you pull constants through integrals?
   i.e. does $\int_a^b (c\cdot f(x))\,dx = c\cdot\left(\int_a^b f(x)\,dx\right)$? Yes No

   (c) Can you distribute integrals across differences?
   i.e. does $\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$? Yes No

   (d) Can you distribute integrals across products or fractions? Yes No!

limits are backward → $\boxed{\int_3^4 f(x)\,dx = -2}$

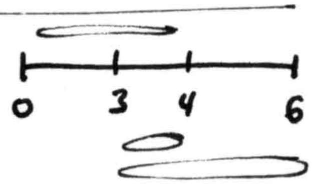


7. Suppose that $\int_0^3 f(x)\,dx = 5$, $\int_4^3 f(x)\,dx = 2$, and $\int_4^6 f(x)\,dx = 10$. Find

   (a) $\int_0^4 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^4 f(x)\,dx = 5 + \left(-\int_4^3 f(x)\,dx\right)$
   $= 5 - 2 = 3$

   (b) $\int_3^6 f(x)\,dx = \int_3^4 f(x)\,dx + \int_4^6 f(x)\,dx = (-2) + 10 = 8$

   (c) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^4 f(x)\,dx + \int_4^6 f(x)\,dx = 5 - 2 + 10$
   $= 13$



8. Suppose that $\int_0^4 f(x)\,dx = 5$, $\int_3^4 f(x)\,dx = -2$, and $\int_3^6 f(x)\,dx = 10$. Find

   (a) $\int_0^3 f(x)\,dx = \int_0^4 f(x)\,dx - \int_3^4 f(x)\,dx = 5-(-2) = 7$

   (b) $\int_4^6 f(x)\,dx = \int_3^6 f(x)\,dx - \int_3^4 f(x)\,dx = 10-(-2) = 12$

   (c) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 7 + 10 = 17$
   ↑ part (a)

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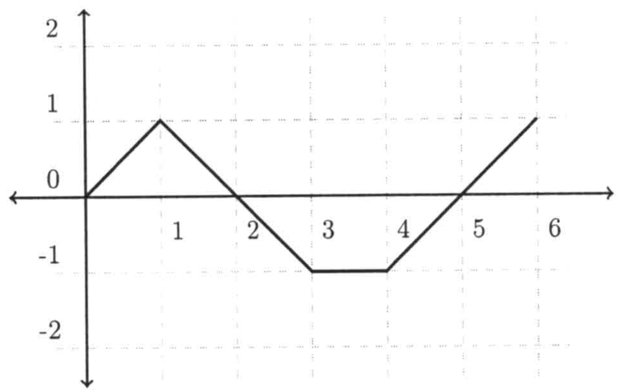
The Fundamental Theorems

1. Compute $\frac{d}{dx}\left[\int_5^x t^2 + 1\,dt\right]$. $= x^2+1$

2. Compute $\frac{d}{dx}\left[\int_1^x \sin(5t)\,dt\right]$. $= \sin(5x)$

3. Compute $\frac{d}{dx}\left[\int_{-3}^x \sin(\cos(e^t))\,dt\right]$. $= \sin(\cos(e^x))$

4. Suppose that the function $f(x)$ is given by the following graph.



Let $A(x) = \int_0^x f(t)\,dt$. Compute the following

(a) $A(1) = \frac{1}{2}$

(b) $A(2) = 1$

(c) $A(4) = -\frac{1}{2}$

$A'(x) = \frac{d}{dx}\left[\int_0^x f(t)\,dt\right] = f(x)$

(d) $A'(1) = 1$

(e) $A'(2) = 0$

(f) $A'(4) = -1$

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Computing Integrals Quickly

1. $\int [\sin(x) + e^x - 2\cos(x) + 6x^2]\, dx$

$= \int \sin(x)\,dx + \int e^x\,dx - 2\int \cos(x)\,dx + 6\int x^2\,dx$

$= -\cos(x) + e^x - 2\cdot\sin(x) + 6\cdot\frac{x^3}{3} + C$

$= -\cos(x) + e^x - 2\sin(x) + 2x^3 + C$

2. $\int_0^1 \frac{1}{x^2+1}\,dx = \tan^{-1}(x)\Big]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$

$\tan^{-1}(1) = x \Leftrightarrow 1 = \tan(x) \Leftrightarrow x = \frac{\pi}{4}$

$\tan^{-1}(0) = x \Leftrightarrow 0 = \tan(x) \Leftrightarrow x = 0$

$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$

3. $\int \frac{x^2+x+1}{x}\,dx = \int \frac{x^2}{x}\,dx + \int \frac{x}{x}\,dx + \int \frac{1}{x}\,dx$

$= \int x\,dx + \int 1\,dx + \int \frac{1}{x}\,dx$

$= \frac{x^2}{2} + x + \ln|x| + C$

4. $\int_{-1}^1 (x^2+3)(x-1)\,dx = \int_{-1}^1 x^3 + 3x - x^2 - 3\,dx$

$= \frac{x^4}{4} + \frac{3x^2}{2} - \frac{x^3}{3} - 3x\Big]_{-1}^1 = \left(\frac{1}{4} + \frac{3}{2} - \frac{1}{3} - 3\right) - \left(\frac{1}{4} + \frac{3}{2} + \frac{-1}{3} + (+3)\right)$

$= \cancel{\frac{1}{4} - \frac{1}{4}} + \cancel{\frac{3}{2} - \frac{3}{2}} - \frac{1}{3} - \frac{1}{3} - 3 - 3$

$= -\frac{2}{3} - 6$

5. $\int \frac{x^{3/2} + \sqrt{x} + 1}{\sqrt{x}}\,dx$

$= \int \frac{x^{\frac{3}{2}}}{\sqrt{x}}\,dx + \int \frac{\sqrt{x}}{\sqrt{x}}\,dx + \int \frac{1}{\sqrt{x}}\,dx$

$= \int \frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}}\,dx + \int 1\,dx + \int x^{-\frac{1}{2}}\,dx$

$= \int x^{(\frac{3}{2}-\frac{1}{2})}\,dx + \int 1\,dx + \int x^{-\frac{1}{2}}\,dx$

$= \int x\,dx + x + \frac{x^{\frac{1}{2}}}{\frac{1}{2}}$

$= \frac{x^2}{2} + x + 2\sqrt{x} + C$

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6. $\int x\sqrt{x^2+3}\,dx$

$u = x^2+3$
$\frac{du}{dx} = 2x$
$du = 2x\,dx$
$\frac{du}{2} = x\,dx$

$= \int \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{2}}\,du$

$= \frac{1}{2}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{u^{\frac{3}{2}}}{3} + C$

$= \frac{(x^2+3)^{\frac{3}{2}}}{3} + C$

7. $\int \frac{1}{x+1}\,dx$

$u = x+1$
$\frac{du}{dx} = 1$
$du = dx$

$= \int \frac{1}{u}\,du = \ln|u| + c$

$= \ln|x+1| + C$

8. $\int \sin(5x)+1\ dx$ $= \int \sin(5x)\,dx + \int 1\,dx$

$u = 5x$
$\frac{du}{dx} = 5$
$\frac{du}{5} = dx$

$= \int \sin(u)\cdot\frac{du}{5} + x$

$= \frac{\cos(u)}{5} + x + C$

$\boxed{= \frac{\cos(5x)}{5} + x + C}$

9. $\int \cos(x)\cdot e^{\sin(x)}\ dx$

$u = \sin(x)$
$\frac{du}{dx} = \cos(x)$
$du = \cos(x)\,dx$

$= \int e^u\,du = e^u + c$

$\boxed{= e^{\cos(x)} + C}$

10. $\int \left[\frac{1}{x^2+1} + \frac{1}{x+1}\right] dx$ $= \int \frac{1}{x^2+1}\,dx + \int \frac{1}{x+1}\,dx$

$u = x+1$
$\frac{du}{dx} = 1$
$du = dx$

$= \tan^{-1}(x) + \int \frac{1}{u}\,du$

$= \tan^{-1}(x) + \ln|u| + c$

$\boxed{= \tan^{-1}(x) + \ln|x+1| + C}$